GREEN'S FUNCTIONS IN THE SOLUTIONS OF HEAT-CONDUCTION PROBLEMS FOR A HOLLOW CYLINDER

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We present solutions of heat-conduction problems for a hollow cylinder for mixed boundary conditions of the second and third kind, the solutions containing rapidly converging series. For the fundamental types of boundary conditions we obtain expressions for the Green's functions which enable us to improve the convergence of the series.

It has already been remarked upon [5] that the application of G. A. Grinberg's method to the solution of heat-conduction problems for a hollow cylinder [3] leads to slowly converging series. In [5] expressions were given for auxiliary functions which make it possible to improve the convergence in many of the cases treated in [3], in particular, in those cases involving mixed boundary conditions of the second and third kind with heat transfer at the outer surface. The same result can be obtained if, following [2], we express the auxiliary function ω in terms of the Green's function $g(r, \xi)$ corresponding to the boundary-value problems:

$$\omega = -\frac{g(r, R_1)\, q_1(\tau)}{\lambda g'_r(R_1+0, R_1)} + \frac{g(r, R_2)\, h_2\psi_2(\tau)}{g'_r(R_2-0, R_2) + h_2 g(R_2-0, R_2)}, \tag{1}$$

where

$$g(r, \xi) = \begin{cases} \dfrac{1}{a}\left(\ln\dfrac{\xi}{R_1} - \dfrac{1}{\mathrm{Bi}_2} - \ln k\right), & r<\xi, \\ \dfrac{1}{a}\left(\ln\dfrac{r}{R_1} - \dfrac{1}{\mathrm{Bi}_2} - \ln k\right), & r>\xi. \end{cases} \tag{2}$$

This way of representing the auxiliary function has the advantage that the series for ω can be obtained without going to the direct calculation of the Fourier coefficients since the Fourier series expansion of the Green's function is already known [2]. Thus we obtain, directly from Eq. (1),

$$\omega = \sum_{n=1}^{\infty}\left\{\frac{2q_1(\tau)\, W_0(\eta_n)}{\lambda g'_r(R_1+0, R_1)} - \frac{2h_2\psi_2(\tau)\, W_0(\eta_n k)}{g'_r(R_2-0, R_2) + h_2 g(R_2-0, R_2)}\right\}\frac{W_0\left(\eta_n \dfrac{r}{R_1}\right)}{aB_n}, \tag{3}$$

where

$$B_n = (k^2\eta_n^2 + \mathrm{Bi}_2^2)\, W_0^2(\eta_n k) - \frac{4}{\pi^2}.$$

Subtracting from this result the series (3) obtained in [3], and then adding the expression (1), we obtain the expression

$$t = -\left(\ln\frac{r}{R_1} - 1/\mathrm{Bi}_2 - \ln k\right)\frac{R_1 q_1(\tau)}{\lambda} + \psi_2(\tau) + \sum_{n=1}^{\infty}\left\{\frac{\eta_n^2}{R_1^2}\cdot \bar{t}_{\eta_n} - \frac{R_1 q_1(\tau)}{\lambda} W_0(\eta_n) - R_2 h_2\psi_2(\tau)\, W_0(\eta_n k)\right\}\frac{2W_0(\eta_n r/R_1)}{B_n}. \tag{4}$$

Here and in what follows we employ the notation used in [3, 5].

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If we consider the same case as that treated in [6], namely, that involving heat transfer at the outer surface, we obtain the following result:

$$t = -\left(\ln\frac{r}{R_1} + \frac{1}{\mathrm{Bi}_1}\right)\frac{R_2 q_2(\tau)}{\lambda} - \psi_1(\tau) + \sum_{n=1}^{\infty}\left\{\frac{\chi_n^2}{R_1^2}\quad \bar{t}_{\chi_n} - \frac{R_2 q_2(\tau)}{\lambda}\, G_0(\chi_n k) - R_1 h_1 \psi_1(\tau)\, G_0(\chi_n)\right\}\frac{2G_0(\chi_n r/R_1)}{B_n}, \tag{5}$$

where

$$B_n = \frac{4}{\pi^2} - (\chi_n^2 + \mathrm{Bi}_1^2)\, G_0^2(\chi_n).$$

In obtaining the expression (5) we have used the Green's function in the form

$$g(r, \xi) = \begin{cases} \dfrac{1}{a}\left(\ln\dfrac{r}{R_1} + \dfrac{1}{\mathrm{Bi}_1}\right), & r < \xi, \\ \dfrac{1}{a}\left(\ln\dfrac{\xi}{R_1} + \dfrac{1}{\mathrm{Bi}_1}\right), & r > \xi. \end{cases} \tag{6}$$

If the initial conditions are represented in the form of a uniformly convergent series, then the series (4) and (5) converge no more slowly than $1/\eta_n^2$ and $1/\chi_n^2$, respectively. Here η_n and χ_n are the roots of the transcendental equations

$$\frac{W_0(k\eta_n)}{W_1(k\eta_n)} = \frac{Y_1(\eta_n)\, J_0(k\eta_n) - J_1(\eta_n)\, Y_0(k\eta_n)}{Y_1(\eta_n)\, J_1(k\eta_n) - J_1(\eta_n)\, Y_1(k\eta_n)} = \frac{k\eta_n}{\mathrm{Bi}_2}, \tag{7}$$

$$\frac{G_0(\chi_n)}{G_1(\chi_n)} = \frac{Y_1(k\chi_n)\, J_0(\chi_n) - J_1(k\chi_n)\, Y_0(\chi_n)}{Y_1(k\chi_n)\, J_1(\chi_n) - J_1(k\chi_n)\, Y_1(\chi_n)} = -\frac{\chi_n}{\mathrm{Bi}_1}. \tag{8}$$

Calculated values of the roots η and χ are given in Table 1.

We also present expressions for the Green's functions for problems with boundary conditions of other types. For mixed boundary conditions of the first and the second kind, the Green's function has the form

$$g(r, \xi) = \begin{cases} \dfrac{1}{a}\left(-\dfrac{\ln\frac{\xi}{R_1}}{\ln k} + 1\right), & r < \xi, \\ \dfrac{1}{a}\left(-\dfrac{\ln\frac{r}{R_1}}{\ln k} + 1\right), & r > \xi, \end{cases} \tag{9}$$

if the boundary conditions at the outer surface are of the first kind, and

$$g(r, \xi) = \begin{cases} \dfrac{1}{a}\ln\dfrac{r}{R_1}, & r < \xi, \\ \dfrac{1}{a}\ln\dfrac{\xi}{R_1}, & r > \xi, \end{cases} \tag{10}$$

when they are of the second kind there.

For mixed boundary conditions of the first and of the third kind, the Green's function has the form

$$g(r, \xi) = \begin{cases} \dfrac{1}{a}\cdot\dfrac{\ln k - \ln\frac{\xi}{R_1}}{\mathrm{Bi}_1 \ln k + 1}\left(\mathrm{Bi}_1 \ln\dfrac{r}{R_1} + 1\right), & r < \xi, \\ \dfrac{1}{a}\cdot\dfrac{\ln k - \ln\frac{r}{R_1}}{\mathrm{Bi}_1 \ln k + 1}\left(\mathrm{Bi}_1 \ln\dfrac{\xi}{R_1} + 1\right), & r > \xi, \end{cases} \tag{11}$$

for boundary conditions of the first kind at the outer surface, and

$$g(r, \xi) = \begin{cases} \dfrac{1}{a}\ln\dfrac{r}{R_1}\left(\dfrac{\mathrm{Bi}_2 \ln\frac{\xi}{R_1}}{1 + \mathrm{Bi}_2 \ln k} + 1\right), & r < \xi, \\ \dfrac{1}{a}\ln\dfrac{\xi}{R_1}\left(\dfrac{\mathrm{Bi}_2 \ln\frac{r}{R_1}}{1 + \mathrm{Bi}_2 \ln k} + 1\right), & r > \xi, \end{cases} \tag{12}$$

for boundary conditions of the third kind at the outer surface.

TABLE 1. Values of the Roots of the Transcendental Equations (7) and (8)

k	Bi						
	0,2	0,5	1	2	3	4	5
				η_1			
1.5	0,5596	0,8706	1,1994	1,6137	1,8861	2,0846	2,2372
1.7	0,4540	0,7038	0,9642	1,2842	1,4882	1,6328	1,7414
2.0	0,3594	0,5551	0,7560	0,9964	1,1446	1,2469	1,3218
2.5	0,2709	0,4168	0,5642	0,7357	0,8379	0,9065	0,9556
3.0	0,2191	0,3363	0,4535	0,5877	0,6660	0,7177	0,7543
				η_2			
1.5	6,3634	6,4248	6,5247	6,7142	6,8896	7,0509	7,1987
1.7	4,5711	4,6246	4,7110	4,8732	5,0205	5,1531	5,2720
2.0	3,2269	3,2715	3,3433	3,4762	3,5943	3,6983	3,7894
2.5	2,1802	2,2149	2,2706	2,3723	2,4605	2,5363	2,8011
3.0	1,6550	1,6834	1,7287	1,8107	1,8809	1,9402	1,9902
				χ_1			
1.5	0,5566	0,8595	1,1707	1,5448	1,7779	1,9408	2,0619
1.7	0,4499	0,6889	0,9266	1,1982	1,3579	1,4649	1,5417
2.0	0,3541	0,5359	0,7090	0,8947	0,9970	1,0624	1,1079
2.5	0,2642	0,3933	0,5090	0,6237	0,6823	0,7180	0,7422
3.0	0,2116	0,3106	0,3953	0,4744	0,5130	0,5358	0,5510
				χ_2			
1.5	6,3834	6,4730	6,6146	6,8704	7,0925	7,2850	7,4522
1.7	4,5951	4,6814	4,8147	5,0448	5,2331	5,3878	5,5159
2.0	3,2546	3,3359	3,4569	3,6524	3,8001	3,9135	4,0024
2.5	2,2106	2,2836	2,3859	2,5360	2,6376	2,7095	2,7624
3.0	1,6859	1,7513	1,8381	1,9553	2,0284	2,0773	2,1120

Similarly, the expressions for the Green's functions when the boundary conditions are of the same kind at both surfaces (conditions of the first, second, or third kind, respectively) are given by:

$$g(r,\ \xi)=\begin{cases} g_1(r)\,g_2(\xi)=\dfrac{\ln\dfrac{r}{R_1}}{a\ln k}\ln\dfrac{R_2}{\xi}, & r<\xi,\\[2ex] g_1(\xi)\,g_2(r)=\dfrac{\ln\dfrac{\xi}{R_1}}{a\ln k}\ln\dfrac{R_2}{r}, & r>\xi,\end{cases} \tag{13}$$

$$g(r,\ \xi)=\begin{cases} \dfrac{r^2+\xi^2}{2a(R_2^2-R_1^2)}-\dfrac{R_1^2\ln\dfrac{r}{R_1}}{a(R_2^2-R_1^2)}-\dfrac{R_2^2\ln\dfrac{\xi}{R_1}}{a(R_2^2-R_1^2)}, & r<\xi,\\[2ex] \dfrac{r^2+\xi^2}{2a(R_2^2-R_1^2)}-\dfrac{R_2^2\ln\dfrac{r}{R_1}}{a(R_2^2-R_1^2)}-\dfrac{R_1^2\ln\dfrac{\xi}{R_1}}{a(R_2^2-R_1^2)}, & r>\xi,\end{cases} \tag{14}$$

$$g(r,\ \xi)=\begin{cases} \dfrac{\ln\dfrac{r}{R_1}+\dfrac{1}{\mathrm{Bi}_1}}{a(1/\mathrm{Bi}_1+1/\mathrm{Bi}_2+\ln k)}\left(1/\mathrm{Bi}_2+\ln k-\ln\dfrac{\xi}{R_1}\right), & r<\xi,\\[2ex] \dfrac{\ln\dfrac{\xi}{R_1}+\dfrac{1}{\mathrm{Bi}_1}}{a(1/\mathrm{Bi}_1+1/\mathrm{Bi}_2+\ln k)}\left(1/\mathrm{Bi}_2+\ln k-\ln\dfrac{r}{R_1}\right), & r>\xi.\end{cases} \tag{15}$$

With the help of the method presented in [2] and use of the expressions (9)-(15), we can obtain auxiliary functions, which lead to series of the types (4) and (5) with a fairly rapid convergence. The roots of some transcendental equations, corresponding to the cases considered, are contained in [1, 4, 7].

NOTATION

R_1, R_2 are the internal and external radii of the cylinder;

$k = R_2/R_1$;

$Bi_1 = \alpha_1 R_1/\lambda$;
$Bi_2 = \alpha_2 R_2/\lambda$;
a is the thermal diffusivity;
$h_1 = \alpha_1/\lambda$;
$h_2 = \alpha_2/\lambda$.

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